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Techniques for Solving a Fractional Definite Integral

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***Abstract*: In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we solve a fractional definite integral by using some methods. In addition, our result is a generalization of classical calculus result.**

***Keywords*: Jumarie's modified R-L fractional derivative, new multiplication, fractional analytic functions, fractional definite integral.**

I. INTRODUCTION

Fractional calculus studies the so-called fractional integral and derivative of real or complex order and their applications. It originated in 1695, in a letter written by L'Hospital to Leibniz, some problems are proposed, such as "what does fractional derivative mean?" or "what is the 1⁄2 derivative of a function?" In the 18th and 19th centuries, many outstanding scientists focused their attention on this problem. For example, Euler, Laplace, Fourier, Abel, Liouville, Grünwald, Letnikov, Riemann, Laurent, or Heaviside. In the past few decades, fractional calculus has been applied to many fields, such as mechanics, economics, viscoelasticity, biology, control theory, and electrical engineering [1-15].

However, the definition of fractional derivative is not unique. Commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie's modified R-L fractional derivative [16-20]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we solve the following α-fractional definite integral:

$$\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{8}\right]^{\frac{1}{\alpha}}}\right)\left[\,Ln_\alpha\left(1+tan_\alpha(x^\alpha)\right)\right],$$

where $0<\alpha\leq 1$. In fact, our result is a generalization of traditional calculus result.

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper.

Definition 2.1 ([21])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

In the following, some properties of Jumarie type of R-L fractional derivative are introduced.

Proposition 2.2 ([22])**:** *If* α, β, x_0, c *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[c] = 0. \tag{4}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([23])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([24])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \tag{6}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ & = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ & = \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ & = \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ & = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{8}$$

Definition 2.5 ([25])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{9}$$

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$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha n}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha\big(g_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha n}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\big(f_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{b_n}{n!}\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}. \tag{12}$$

Definition 2.6 ([26])**:** Let $0 < \alpha \le 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} x^\alpha. \tag{13}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.7 ([27])**:** If $0 < \alpha \le 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha n}. \tag{14}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha 2n}, \tag{15}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{16}$$

Definition 2.8 ([28])**:** Let $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha m} = f_\alpha(x^\alpha) \otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the mth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha (-1)}$.

Definition 2.9 ([29])**:** Let $0 < \alpha \le 1$ and r be a real number. The r-th power of the α-fractional analytic function $f_\alpha(x^\alpha)$ is defined by

$$[f_\alpha(x^\alpha)]^{\otimes_\alpha r} = E_\alpha\Big(r \cdot Ln_\alpha\big(f_\alpha(x^\alpha)\big)\Big). \tag{17}$$

Definition 2.10 ([30])**:** The smallest positive real number T_α such that $E_\alpha(iT_\alpha) = 1$, is called the period of $E_\alpha(ix^\alpha)$.

Proposition 2.11 ([31])**:** *If $0 < \alpha \le 1$, p, q are real numbers, $f_\alpha(x^\alpha)$ is a α-fractional analytic function, then*

$$\left({}_pI_q^\alpha\right)[f_\alpha(x^\alpha)] = \left({}_pI_q^\alpha\right)\left[f_\alpha\left(\frac{1}{\Gamma(\alpha+1)} p^\alpha + \frac{1}{\Gamma(\alpha+1)} q^\alpha - \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)\right]. \tag{18}$$

III. MAIN RESULT

In this section, we will solve a fractional definite integral.

Theorem 3.1: *If $0 < \alpha \le 1$, then the α-fractional definite integral*

$$\left({}_0I^\alpha_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha\big(1 + tan_\alpha(x^\alpha)\big)\right] = \frac{T_\alpha}{16} \cdot Ln_\alpha(2). \tag{19}$$

Proof Since

$$\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha\left(1+tan_\alpha(x^\alpha)\right)\right]$$

$$=\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha\left(1+tan_\alpha\left(\frac{T_\alpha}{8}-\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\right)\right] \quad \text{(by Proposition 2.11)}$$

$$=\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha\left(1+\left[tan_\alpha\left(\frac{T_\alpha}{8}\right)-tan_\alpha(x^\alpha)\right]\otimes_\alpha\left[1+tan_\alpha\left(\frac{T_\alpha}{8}\right)\otimes_\alpha tan_\alpha(x^\alpha)\right]^{\otimes_\alpha(-1)}\right)\right]$$

$$=\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha\left(1+[1-tan_\alpha(x^\alpha)]\otimes_\alpha[1+tan_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right)\right]$$

$$=\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha\left(2\otimes_\alpha[1+tan_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right)\right]$$

$$=\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha(2)-Ln_\alpha\left(1+tan_\alpha(x^\alpha)\right)\right]$$

$$=\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{8}\right]^{\frac{1}{\alpha}}}\right)[Ln_\alpha(2)]-\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha\left(1+tan_\alpha(x^\alpha)\right)\right]$$

$$=\frac{T_\alpha}{8}\cdot Ln_\alpha(2)-\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha\left(1+tan_\alpha(x^\alpha)\right)\right]. \tag{20}$$

It follows that

$$\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha\left(1+tan_\alpha(x^\alpha)\right)\right]=\frac{T_\alpha}{16}\cdot Ln_\alpha(2).$$

Q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional derivative and a new multiplication of fractional analytic functions, we solve a fractional definite integral by using some techniques. On the other hand, our result is a generalization of traditional calculus result. In the future, we will continue to use Jumarie type of R-L fractional calculus and the new multiplication of fractional analytic functions to solve the problems in engineering mathematics and fractional differential equations.

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